The Twin Paradox

Presented By:

Bob Coulson
Tyler Stelzer
Berit Rollay
A.J. Schmucker
Scott McKinney
Overview

- Events and Coordinatizations; The concept of Spactime
- Lorentz Coordinatizations; Lorentz Postulates
- Minkowski Space
- Moving Reference Frames
  - Time Dilation
  - Light Scales
  - Length Contraction
- Lorentz Transformations
- Lorentz-Einstein Transformations
  - Boosts
- The Twin Paradox
**Events and Coordinizations:**
**The Concept of Spacetime**

*Events: What are they?*

An “event” is a definite happening or occurrence at a definite place and time. A couple of examples of events are a bomb exploding or an emission of a photon, (particle of light), by an atom which is similar to switching a light on and off.

*What is Spacetime?*

Let $E$ be the set of all events as previously defined. This set is call **Spacetime**. Let $e$ represent a particular event, then “$e \in E$” means that $e$ is an event.

*Modeling Spacetime*

To model spacetime we use $\mathbb{R}^4$. The idea behind this is that each event $e \in E$ is assigned a coordinate $(T_e, X_e, Y_e, Z_e) \in \mathbb{R}^4$. $\mathbb{R}^4$ is an ordered four-tuple. This means that everything in $\mathbb{R}^4$ has 4 coordinates. For example: $(x,y,z,w)$ has 4 coordinates.

*What Are $T_e$, $X_e$, $Y_e$, $Z_e$?*

- $T_e$ - The time coordinate of the event.
- $X_e$ - The $x$ position coordinate of the event.
- $Y_e$ - The $y$ position coordinate of the event.
- $Z_e$ - The $z$ position coordinate of the event.

*What does this mean?*

- $e \rightarrow (T_e, X_e, Y_e, Z_e)$ is assumed to be a bijection. This means that:
  - It is a one to one and onto mapping
  - Two different events need to occur at either different places or different times.
- Such assignment is called a “**Coordination of Spacetime**”. This can also be called a “**Coordination of E**”.

**Lorentz Coordinizations & Postulates**

*Vector position functions and Worldlines*

- In Newtonian physics/calculus, moving particles are described by functions 
  \[ t \rightarrow r(t) \]
- \[ r(t) = (x(t), y(t), z(t)) \]

This curve gives the “history” of the particle.
- View this from $\mathbb{R}^4$ perspective
  \[ t \rightarrow (t, r(t)) \]
• In the above ‘t’ represents time and ‘r(t)’ represents the position.
• This can be thought of as a “curve in $\mathbb{R}^4$”, called the **Worldline** of the particle.

**What is time?**
• There are 2 types of time
  • Physical Clock Time
  • Coordinate Time: the time furnished by the coordinatization model:
    \[ e \rightarrow (T_e, X_e, Y_e, Z_e) \]

**1st Lorentz Postulate**
• For stationary events, Physical Clock Time and Coordinate Time should agree
• That is, we assume that stationary standard clocks measure coordinate time.

**2nd Lorentz Postulate**
• The velocity of light called $c = 1$.
• Light always moves in straight lines with unit velocity in a vacuum.
• $T | \rightarrow (T, vT + r_0)$, time and spatial position
• Note: Think of the light pulse as a moving particle.

**Minkowski Space**

In Minkowski Space, which could also be thought of as the geometry of space-time; the symmetric, non-degenerate bilinear form of the inner product has the following properties:

• $<x, y> = <y, x>$
• $<x_1 + x_2, y> = <x_1, y> + <x_2, y>$
• $<cx, y> = c<x, y>$
• The inner product does not have to be positive definite, which means the product of it with itself could be negative.

The inner product being non-degenerate means only the zero vector is orthogonal to all other vectors.

Space-time has its own geometry described by the Minkowski Inner Product. The Minkowski Inner Product, defined on $\mathbb{R}^4$, given a four dimensional vector $u$ and a four dimensional vector $v$ is:

\[
\begin{align*}
    u &= (u^0, u^1, u^2, u^3) \\
    v &= (v^0, v^1, v^2, v^3) \\
    <u, v> &= u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3
\end{align*}
\]

The Minkowski Inner Product is also called the Lorentz Metric, the Minkowski metric, and the Metric Tensor. $M = \mathbb{R}^4$ with the Minkowski Inner Product. “•” represents the usual inner product (dot product) in $\mathbb{R}^3$. In this case you have an inner product that allows negative length.
How is the Minkowski Inner Product related to the Euclidean Inner Product? The Euclidean Inner Product given a three dimensional vector \( r \) and a three dimensional vector \( s \) is:

\[
\begin{align*}
  r &= (r^1, r^2, r^3) \\
  s &= (s^1, s^2, s^3) \\
  r \cdot s &= (r^1 s^1 + r^2 s^2 + r^3 s^3)
\end{align*}
\]

Note: \( R^4 = R^1 \times R^3 \)

Whereas, the Minkowski Inner Product given a vector \( u \) made up of a one dimensional time component and a three dimensional spatial component and a vector \( v \) also made up of a one dimensional time component and a three dimensional spatial component is:

\[
\begin{align*}
  u &= (u^0, (u^1, u^2, u^3)) \\
  v &= (v^0, (v^1, v^2, v^3)) \\
  <u, v> &= u^0 v^0 - (u^1, u^2, u^3) \cdot (v^1, v^2, v^3)
\end{align*}
\]

Strange Things Can Happen In Minkowski Space such as:

- Vectors can have “negative lengths”
- Non-Zero vectors can have zero length.

A vector \( v \) an element in \( M \) is called:
- “Time Like” if \(<v, v> > 0\)
- “Null” if \(<v, v> = 0\)
  
  (Some of these are Non-Zero Vectors with zero length.)
- “Space Like” if \(<v, v> < 0\)
  
  (These are the negative length vectors.)

Minkowski Space serves as a mathematical model of space-time once a Lorentz coordinization is specified. Consider an idealized infinite pulse of light.

Consider the Problem of Describing Light. We think of a moving light pulse as a moving particle emitted via a flash in space-time. The path of this particle is referred to as its worldline. By the second Lorentz Postulate, the worldline is given by: \( T \mapsto (T, v T + r_0) \). \( T \) representing time, \( v T + r_0 \) representing the equation of the trajectory of the light pulse, and \( r_0 \) representing the origin of the light pulse. Recall that the dot product of the vector \( v \), which is a three dimensional vector, with itself is equal to one and that \( r_0 \) is also an element of \( R^3 \). \( (T, v T + r_0) = (T, v T) + (0 , r_0) = T(1, v) + (0 , r_0) \) Note: \( T(1, v) \) is a null vector because \(< (1, v) , (1, v)> = 1 - v \cdot v = 1 - 1 = 0\). This is an example of a Non-Zero Vector with zero length. Setting a variable \( a:=(1,v) \) and a variable \( b:=(0, r_0) \) and plugging these variables into our current equation results in the worldline of a light pulse being of the form \( T \mapsto aT+b \) and element in \( M \) with \(<a,a> = 0\). These are called null lines.

In respect to light cones, suppose \( b \) is an element in \( M \). The light cone at \( b:=\{p \in M \mid P - b, P - b > = 0\} \). This is the union of all null lines passing through \( b \). The forward light cone at \( b = \{P=(P^0,P^1,P^2,P^3)\in M \mid P \in \text{light cone at } b, P^0-b^0>0\} \) The backward light cone at \( b = \{P=(P^0,P^1,P^2,P^3)\in M \mid P \in \text{light cone at } b, P^0-b^0<0\} \)
Moving Reference Frames

To understand the idea of moving reference frames, we need to recall the idea of a coordinatization. This was defined as

\[ e \in E \rightarrow (T_e, X_e, Y_e, Z_e) \]

For a visual aid, you can imagine the following:

Looking at the above picture, we want to be able to find out the spatial coordinates \((X_e, Y_e, Z_e)\) of the event labeled “e” (could be thought of as a burst of light). Using some trigonometry, we should be able to do this.

If we assume that

i) \( c = \text{speed of light} \)
ii) \( \text{rate} \times \text{time} = \text{distance} \)

We can derive the following equation:

\[ T_e = T - \frac{\sqrt{X_e^2 + Y_e^2 + Z_e^2}}{c} \]

where \( T \) is the time at which the light pulse reaches \( O \) (the observer positioned at the origin from the graph above).
Now, let’s consider an interesting math problem. Suppose there is a 2\textsuperscript{nd} coordinate system, moving at a constant velocity \( v \), in the direction of the x-axis. Suppose \( O, O' \) both employ the same procedure for coordinatizing E:

In the picture below, the coordinate system

\[(T, X, Y, Z)\] will represent the stationary frame

and

\[(T', X', Y', Z')\] will represent the moving frame.

How are these two frame related? First, let’s look at a picture.

To mathematically find out how these two frame are related, we’ll have to assume the following:

i) \( O, O' \) both have standard clocks. An example of a standard clock would be an Einstein Langevin clock (think of this as a light pendulum).

ii) \( O \) and \( O' \) have a rigid rod or tube of length \( L \). See Picture below.

Looking at the above picture, one unit of time is the duration of time between emission and return.
Time Dilation

Now that we have two observers, one moving and one stationary, we want to ask the question, how does O (stationary) regard O’’s (moving) clock? To do this, we’ll think of O’’s clock as sitting in a moving vehicle such as a train or spaceship. To assist in visualization, consider the following pictures.

Moving Spaceship (O’s perspective)

![Diagram of a spaceship moving at a velocity v, with a light source and a mirror, and the equation \( L = ct' \).]

Let \( t' \) be the time of \( \frac{1}{2} \) pendulum of O’’s clock as observed by O. Observe: \( (ct)^2 = L^2 + (vt)^2 \)

In this diagram, the mirror has actually moved since the ship has moved.
Using some simple algebra, we want to solve the previously observed equation, $(ct)^2 = L^2 + (vt)^2$, for $t$. So

\[ c^2t^2 = L^2 + v^2t^2 \]
\[ c^2t^2 - v^2t^2 = L^2 \]
\[ t^2(c^2 - v^2) = L^2 \]
\[ t^2 = \frac{L^2}{c^2 - v^2} \]
\[ t = \frac{L}{\sqrt{c^2 - v^2}} \]

If we let $c = 1$

\[ t = \frac{t'}{\sqrt{1-v^2}} \]
\[ t = \gamma(v)t' \]

(Note: $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$)

Now that we have some framework laid out, we want to suppose the “rod” is situated in the direction of the moving frame. Looking at a picture, this would give us:

**Light Scales**

Consider a “rod” of length $R$. Since we assume $(\text{Rate})(\text{Time}) = \text{Distance}$, the length $R$ may be measured by light rays as follows. Using the same diagram as before, we have the following:

where

i) $c = \text{speed of light} = 1$ (for our purposes. Speed of light is actually 186,000 miles per second)

ii) $(c)(\text{time}) = 2R$ ($R$ is the length of the Rod)

iii) $R = (\text{time}) / 2$ or $R = t / 2$

Now that we have some framework laid out, we want to suppose the “rod” is situated in the direction of the moving frame. Looking at a picture, this would give us:
Looking at the above picture, we’ll let $R' = \frac{R}{2}$. From O’s perspective the rod is in motion, as O’s light scale functions.

So, if we let:

i) $t_1$ = time on O’s clock until the flash reaches the mirror.
ii) $t_2$ = time the flash takes between the mirror and returning to the light source.

So, the total time (on O’s clock) is

$$t = t_1 + t_2$$

Now that we have an equation for the total time, we need to figure out what $t_1$ and $t_2$ are. To do this, we’ll look at another picture. From the picture below:

$Vt_1$ = distance the spaceship (and hence rod) travels between initial flash and mirror interception.

From this picture, we can derive an equation for $t_1$ as follows.

$$ct_1 = R + vt_1$$
$$t_1 = R + vt_1$$
$$t_1 - vt_1 = R$$
$$t_1(1 - v) = R$$
$$t_1 = \frac{R}{1 - v}$$

Now that $t_1$ is done, the next step is to figure out an equation for $t_2$. To do this, we will look at another picture.
Using some simple algebra, we can solve for $t_2$ as we did for $t_1$ as follows:

$$R = ct_2 + vt_2$$
$$R = t_2 + vt_2$$

$$t_2 = \frac{R}{1+v}$$

Now that we have equations for $t_1$ and $t_2$, we can revisit the total time equation $t = t_1 + t_2$. Substituting in for $t_1$ and $t_2$, we get the following:

$$t = t_1 + t_2$$
$$t = \frac{R}{1-v} + \frac{R}{1+v}$$
$$t = R\left(\frac{1}{1-v} + \frac{1}{1+v}\right)$$

Lorentz Fitzgerald Contraction

The above equations build the framework for Lorentz Fitzgerald Contractions.

Earlier, we stated that:

$$R' = \frac{t'}{2} \quad \text{and} \quad t = \frac{2R}{1-v^2}$$

Also,

Recall: $t = \gamma(v)t'$ \quad \left(\gamma(v) = \frac{1}{\sqrt{1-v^2}}\right)$
So, replacing values for t, we get:

\[
\frac{2R}{1-v^2} = \frac{1}{\sqrt{1-v^2}} (2R')
\]

\[
\frac{R}{1-v'^2} = \frac{R'}{\sqrt{1-v^2}}
\]

\[
\frac{\sqrt{1-v^2}}{1-v'^2} (R) = R'
\]

and we come up with the following equations that define the Lorentz Fitzgerald Contraction

\[
R' = \gamma(v) R \\
t = \gamma(v) t'
\]

**Lorentz Transformations**

If we are to model space time with a four-variable coordinate system, as in \( \mathbb{R}^4 \), then we can consider a coordinatization of an event \( e \) to be: \( (t,x,y,z) \). Next, we will consider a new coordinatization of this event and label it as: \( (t',x',y',z') \). What we want to do is consider the relationship between these two coordinatizations by assuming that the mapping from \( e \) to \( e' \) is a bijection. We can now ask ourselves what bijections will preserve the properties of Lorentz coordinatizations. Remember that in a Lorentz frame, the worldlines of light pulses are exactly the null lines in \( \mathbb{M} \). These worldlines are described as follows: \( (at + b, \text{where } a, b \in \mathbb{M} \text{ and } <a,a> = 0) \). Hence, a Lorentz preserving bijection must map these null lines to other null lines. Therefore, we can look at what transformations preserve the null line condition. \( (<a,a> = 0) \). The transformations of interest to us are the Minkowski Metric preserving linear maps. If we call the transformation \( L \), and pick two vectors \( U \), and \( V \), then these are the bijections in which:

\[
<L,U,V> = <U,V>
\]

We can show that these transformations preserve null lines by the following proof:

\[
L(at + b) = L(at) + L(b) = tL(a) + L(b).
\]

\[
<L(a),L(a)> = <a,a> = 0.
\]
These transformations are called Lorentz Transformations.

**Lorentz – Einstein Transformations (Boosts)**

We now want to come up with a way to relate coordinatizations given to a single event by observers in separate reference frames. We will consider the case that one reference frame is stationary (relative to itself, called O), and the second frame moves (relative to the first frame, called P) along the x-axis. An event, (e), which occurs on the x-axis of both frames is given coordinates (t,x,y,z) by the stationary frame (O), and (t’,x’,y’,z’) by the moving frame (P). Since distance equals rate multiplied by time (D = RT), then the x-coordinate of (P) from (O)’s perspective is (vt), velocity by time. So therefore, the distance between (e) and (P) from (O)’s perspective is (x-vt). Since (O) understands that (P) will measure this distance as a length contraction, then we can apply the contraction developed earlier, where:

\[
R' = \gamma(v)R
\]

And therefore:

\[
x' = \gamma(v)(x - vt)
\]

We now consider the time element of the coordinatizations. We are going to let (t’o) be the time that the event (e) reaches (P). Also, we let (t*) be this exact time as measured by (O)’s clock. We can note here that (t*) is greater than (t), the original time of (e), and that the time (t*-t) will be the transit time from (e) to (P) as measured by (O)’s clock.

The following diagram illustrates the distances between (O), (P), and (P) as measured by (O) (the dotted frame).
Since (O) will measure a longer time than (P), the distance (VT*) will be greater than (VT). All measured distances here are from the point of view of (O). So now, again, since distance = rate * time, then:

\[ c(t^*-t) = x - vt \]

Here we assign the speed of light (c) to be 1, and therefore:

\[ t^*-t = x - vt \]

Now we solve for the variable (t*):

\[
\begin{align*}
  t^* + vt^* &= x + t \\
  t^*(1 + v) &= x + t \\
  t^* &= \frac{x + t}{1 + v}
\end{align*}
\]

We now can transform (t*) into the time as measured by (P) by adjusting for time dilation using the formula developed earlier, where:

\[ t = \gamma(v)t' \]

Then, substituting in for (t*) as the stationary time we get:

\[
\frac{(x + t)}{(1 + v)} = \gamma(v)t'_o
\]

and since:

\[ \gamma(v) = \frac{1}{\sqrt{1 - v^2}} \]

Then:

\[
\begin{align*}
  \frac{(x + t)}{(1 + v)} &= \frac{t'_o}{\sqrt{1 - v^2}} \\
  t'_o &= \sqrt{1 - v^2} \frac{(x + t)}{(1 + v)}
\end{align*}
\]

Now, finally, we can recall the transformation of the length, (x), coordinate to solve for the time as measured by (P):
First, recall:

\[ x' = \gamma(v)(x - vt) \quad \text{And} \quad t' = x' + t \]

Then substituting for \((x')\):

\[ \sqrt{1 - v^2} \frac{x + t}{1 + v} = \gamma(v)(x - vt) + t' \]

Finally, solving for \((t')\):

\[
\begin{align*}
&= [\sqrt{1 - v^2}(x + t)/(1 + v)] - (x - vt)/\sqrt{1 - v^2} \\
&= [\sqrt{1 - v(x + t)}/\sqrt{1 + v}] - (x - vt)/\sqrt{1 - v}\sqrt{1 + v} \\
&= [(1 - v)(x + t) - (x - vt)]/\sqrt{1 - v^2} \\
&= (t - vx)/\sqrt{1 - v^2} \\
&= \gamma(v)(t - vx)
\end{align*}
\]

We now have the time as well as the \(x\) – axis component of the relation. Since the moving frame is traveling along the \(x\) – axis of the stationary frame, it is easy to see that the \(y\) and \(z\) components will not change. Therefore the final transformation looks as follows:

\[
\begin{align*}
& t' = \gamma(v)(t - xv) \\
& x' = \gamma(v)(x - vt) \\
& y' = y \\
& z' = z
\end{align*}
\]

This transformation directly relates the coordinatization of the \(e\) between a stationary frame of reference and one moving along it’s \(x\) – axis at a fixed velocity \(v\). This is also called a boost in the \(x\) – direction.

We can now ask ourselves this question: Is this transformation a Lorentz Transformation? The second postulate of Lorentz coordinatizations tells us that a Lorentz transformation of a Lorentz coordinatization will produce another Lorentz coordinatization. Showing the boost transformation is Lorentz will support this hypothesis.
To show that the transformation is indeed Lorentz, we need to show that the mapping preserves the Minkowski inner product of the original coordinates. In other words, if we assume some original vectors $A$ and $B$, we must see that:

$$< A, B > = < L A, L B >$$

We must expand the inner product of both sides to see that they are equal:

$$< a, b > = t_a t_b - x_a x_b - y_a y_b - z_a z_b$$

$$< L a, L b > = \gamma(v)(t_a - v x_a)\gamma(v)(t_b - v x_b)$$

$$- \gamma(v)(t_a - v x_a)\gamma(v)(t_b - v x_b)$$

$$- y_a y_b - z_a z_b$$

$$= \gamma(v)^2(t_a t_b + x_a x_b v^2) - \gamma(v)^2(x_a x_b + t_a t_b v^2)$$

$$= \gamma(v)^2(t_a t_b + x_a x_b v^2 - x_a x_b - t_a t_b v^2)$$

I have simplified the process here by leaving out the quantity $(-y_a y_b - z_a z_b)$, we can see that these terms already match the terms of the original vectors. Also, recall that:

$$\gamma(v) = 1/\sqrt{1 - v^2}, \text{ So } \gamma(v)^2 = 1/(1 - v^2)$$

Therefore, substituting in for the gamma function, the equation becomes:

$$[1/(1 - v^2)](t_a t_b - t_a t_b v^2 - x_a x_b + x_a x_b v^2)$$

$$= 1/(1 - v^2)[(t_a t_b (1 - v^2) - x_a x_b (1 - v^2))]$$

$$= t_a t_b - x_a x_b$$

Thus we have shown that the boost transformation is a Lorentz transformation.
The Twin Paradox
(An application of the time dilation)

The idea with these examples is that due to Special Relativity and the idea of time dilation, if two twins are born and one stays on Earth while the other boards a space ship and rockets away at a fraction of the speed of light then the twin on board the space ship will age more slowly than the one on the planet based on the speed of the ship. Here are some equations and variables to keep in mind.

Recall:

\[ t = \frac{t'}{\sqrt{1-v^2}} \]
\[ t = \gamma(v)t' \]

(Note : \( \gamma(v) = \frac{1}{\sqrt{1-v^2}} \))

Our first example dealing with time dilation deals with this problem. If a space ship’s velocity is equal to .5c, and ten years pass on Earth, how many years would pass on the ship? The following explains the math of the problem.

\[ t' = t\left(1-v^2\right)^{1/2} \]
\[ t' = 10\left(1-.5^2\right)^{1/2} \]
\[ t' = 8.66 \text{ years} \]

To explain the result, if 10 years passes on Earth then only 8.66 years would have passed on the space ship.

Our next example keeps with the theme of a space ship. Let’s say the space ship in Example 1 passes us as its clocks read 12 o’clock. In our reference, how far away will it be when its clocks read 1 pm? If we do these calculations we find our answer.

\[ t' = 10\left(1-.5^2\right)^{1/2}; t = 8.66t \text{ years} \]
\[ 3.35 \times 10^8 \text{ mph} \times 1.155 = 3867 \times 10^8 \text{ miles} \]

Again to explain, if one hour passes on the ship then 1.155 hours passes on Earth. Since the ship is moving at .5c or 3.35*10^8 mph. The ship would be 3.867*10^8 miles away when its clocks read 1 pm.

Our final example approaches time dilation from the opposite direction. How fast must a space ship travel in order that its occupants will only age 10 years while 100 years passes on Earth? By solving for the variable ‘v’ we find our answer.
\[ t = t' (1 - v^2)^{1/2} \]
\[ 10 = 100 (1 - v^2)^{1/2} \]
\[ v = (-1/100 + 1)^{1/2} = .995 \]

So if a ship is going .995c and 10 years past inside it then 100 years would pass on Earth.

**References**