RADON TRANSFORMS

“More than Meets the Eye”
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Part I: Radon Transforms

Transformers to Radon Transforms; how do these two relate you might be asking.

 Everyone probably remembers the Transformers cartoon when we were kids. The theme of the Transformers was “It is a world transformed, where things are not what they seem…” This was the motto of the shape changing toy robots that most of us probably had as kids. And everyone knew Optimus Prime, on the outside, a large semi, but you never knew what was hidden on the inside until you transform him into the commander of the Autobots.

There are a lot of similarities From Transformers to radon transforms, which we will be talking about today. They are a way of finding out what is on the inside, when all we can see or find out is what is on the outside. For example, the basic principals of Tomography, the three-dimensional image reconstruction used in x-rays and cat-scans, was developed from the pure mathematics of the radon transforms. We are going to walk you through how they’re used to take what you know just by looking at something, and transforming it into what you don’t know and can’t see on the inside.
I’ll now introduce the formula for the radon transform to you, then go into detail later on how it’s constructed and we’ll try to make sense of it’s power and usefulness in applications of today.

\[ R(U,\phi) = \int_{-\infty}^{\infty} F(x,y) \, dv \]

The radon transform was first introduced in 1917 by Johann Radon. He was born in Bohemia, now the Czech republic. Here he studied and later became a professor in Austria before introducing his “Radon Transforms.” Radon’s work, which appeared to have been done for the pure mathematics behind, has played a very important role in the development of Tomography. This is one of the many instances where most look at something and say where am I ever going to use this, but it’s surprising how often it can be extremely useful when applied elsewhere. When he developed this he had no idea how it would be used.

To begin understanding the radon transform, we must first understand the idea of Ray sums. To understand Ray sums, we might want to go over the basic principals of how an x-ray machine works. In an x-ray, the ray is directed through some mass, and the strength of the x-ray after it goes through the mass is marked on a film on the other side of the mass. So in a way it records how much mass gets in the way of the ray on it’s way through the body. Now lets try to put this together with what a ray sum is.
Say we have some large mass, and we are taking an x-ray of just a slice of it. If we could see this actual slice it would look like a density function, \( f(x,y) \) where the mass density is shown at each point \((x,y)\). Now if we were to direct a small x-ray through it, say \( L \). The ray sum would be the total mass along this line \( L \) through this plane. This is considered a set function, something that takes a line, slice, or set of points and produces a number as output.

This mass density function was defined on the \( x,y \) coordinate system. Now is the time to introduce the \( u,v \) coordinate system which is oriented the same as the \( X \) and \( Y \) axes, but has an angle of \( \phi \) between the \( x \) and \( u \) axes. We must have a way of converting form \( x,y \) to \( u,v \). Say we have one point. It is point \((x,y)\) in one system, and \((u,v)\) in the other respectively. It’s easy to see that \( U = R \cos(\theta) \), and \( V = R \sin(\theta) \). In the same way, \( X = R \cos(\phi + \theta) \) and \( Y = R \sin(\phi + \theta) \). Now thinking back to our Trigonometry days and the sum and differences formulas: \( \sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v) \) and \( \cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v) \). Substituting the equations for \( X \) and \( Y \) into these sum and differences formulas, then plugging the equations for \( U \) and \( V \) into that we come up with our desired conversions form \( x,y \) into \( u,v \): \( x = u \cos(\phi) - v \sin(\phi) \) and \( y = v \cos(\phi) + u \sin(\phi) \).

Now that we know and understand what an individual ray sum is, and understand the \( u,v \) coordinated system and how there is just an angle of \( \phi \) between the \( U \) and \( X \) axes we can define the entire Ray sum function. The entire ray sum function consists of all
possible ray sums, at every angle phi from 0 to \( \pi \) and at every distance \( U \) on the \( U \) axis. We have now defined the Ray Sum function for the mass density function, \( f(x,y) \).

Now lets take a step back and see what we now know. An x-ray machine just finds the sum of the mass along any line. We know this, but we still have no idea of how the mass is distributed along this line. Now the question comes to how we can convert the Ray Sum function, something we know, to the mass density function, what we are seeking. The answer is simple, use the Radon Transform.

It is now fairly easy to see that one ray sum along the line \( L \) is equal to the line integral of the mass density function along that line \( L \). Now we can introduce the Radon Transform, which uses the entire ray sum set function. In a normal function you have input(s), a routine is performed, and then you receive output(s). the radon transform takes the actual density function \( f(x,y) \) as an input, and has the set function \( R(U,\phi) \) as it’s output. So the mapping of \( f(x,y) \) into the Ray Sum function, \( R(U,\phi) \) is known as the radon transform. Johann Radon showed that if \( f \) is continuous the \( f(x,y) \) is uniquely determined by the values of \( R(U,\phi) \). He developed the idea of transforming the known values of \( R(U,\phi) \) into the actual density function. This will produce the three dimensional image that we are seeking, the final result of an x-ray. Since we know the output, \( R(U,\phi) \), but not the input, \( f(x,y) \), it now turns into an inverse problem.
Part II: The Concept of an Inverse Problem

As mentioned, the problem of finding the mass-density function of an object is approached as an inverse problem. Consider the following passage from Book VII of Plato’s Republic:

Behold! Human beings living in an underground den; here they have been from their childhood, and have their legs and necks chained so that they cannot move, and can see only before them, being prevented by the chains from moving their heads. Above and behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and you will see, if you look, a low wall built along the way, like the screen which marionette players have in front of them, over which they show their puppets. ¹

Here, Plato describes a situation where a fire in a cave casts the shadows of objects onto a wall. The people in the cave have been chained down so that they cannot see the objects themselves, but only the shadows, creating what has been mentioned as an Inverse Problem. The question remains, can they determine what the objects look like in 3 dimensions given only these shadows?

Before explaining the inverse problem any further, we might consider the direct problem first. The direct problem is much easier solved in this case then the inverse problem. To find the shadows of our hand for instance, we could merely trace our hand onto a piece of paper, approximating the shadows. The corresponding inverse problem is given one of these traces, could we determine what the hand looked like.

It turns out that this inverse problem is impossible to solve with the given amount of data. Since the tracing is only 2 dimensional, there has been a loss of data from the 3rd dimension. The intriguing problem still exists of how to replace the lost dimension.
Tomography solves this by taking such “tracings” from man angles, then combining the results into one 3 dimensional object.

**Part III: Tomography**

Tomography is the science of three-dimensional image reconstruction. It is generally implemented in the medical sciences field to identify mass density irregularities within the human body, such as lesions or tumors. Either one of these examples would register on an x-ray machine, however, it is important to know exactly where the problem exists on the two-dimensional image. The x-ray machine can identify mass density in the 2-dimension, and through tomography a three-dimensional image map can be created based on many two-dimensional photographs taken from different angles.

The goal of the mathematics behind tomography that we studied is to identify the mass density function. This is necessary to identify three-dimensional mass distribution. A region of high mass density could be at any depth or even spread out evenly back from the pictures 2-dimensional face. By finding the mass density function we can find how the density is spread throughout a region back along an imaginary z-axis.

The process to find the mass density function is as follows: Take the given Ray-Sum and plug it into the Fourier Transform, Apply Fourier Inversion on that Fourier Transform, Solve for the mass density function.
**Fourier transforms:**

The first step to finding the density function is developing the Fourier Transform. The

\[ g(w) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i wx} \, dx \]

Fourier Transform \( g \) of \( f \) would be in this notation:

Since Tomography reads in two-dimensional images this Fourier Transform needs to be converted to find the Fourier Transform of \( f(x,y) \). This is the Fourier Notation in two-dimension:

\[ g(w, \sigma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i (wx + \sigma y)} \, dx \, dy \]

In order to find the Mass Density Function we need to implement the Fourier Inversion Theorem which inverts the Fourier Transform to instead solve for \( f(x,y) \):

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(w, \sigma) e^{2\pi i (wx + \sigma y)} \, dw \, d\sigma \]

Now clearly if we can find the Fourier Transform of \( f(x,y) \) we can find the Mass Density Function.

**Ray-Sums:**

The next step in finding the Mass Density Function is to solve for the Fourier Transform given a Ray-Sum. The first step in this process is to change the axes from x-y to u-v.

\[
\cos \phi = \frac{w}{\sqrt{w^2 + \sigma^2}} \quad \sin \phi = \frac{\sigma}{\sqrt{w^2 + \sigma^2}}
\]

These new axis will be shift apart from the x-y by some angle \( \Phi \). Thus \( \Phi \) would have:

For some \( w \) and \( \theta \) describing the rotation amount.

We can solve for \( wx+\sigma y \):

Recall \( x=\sin - v \cos, y= u \cos + v \sin \) for an axis of rotation.
Plug in these values for x and y into wx+oy to get:

\[ wx + \sigma y = w(u \cos \phi - v \sin \phi) + \sigma (u \sin \phi + v \cos \phi) \]

Next solve this using the given cos(\(\Phi\)) and sin(\(\Phi\)) values. For the final answer:

\[ wx + \sigma y = (\sqrt{w^2 + \sigma^2})u \]

Also recall the Ray-Sum that was previously proven:

\[ R(u, \phi) = \int_{-\infty}^{\infty} f(x, y)dy \]

\[ g(w, \sigma) = \int_{-\infty}^{\infty} R(u, \phi)dv e^{-2\pi i (wx + \sigma y)u} du \]

Substitute these two conclusions into the Fourier Transform of \(f(x,y)\):

Now we have an equation that can be solved given a Ray-Sum. Once this equation is solved we can implement the Fourier Inversion Theorem and use the Fourier Transform to solve for the Mass Density Function. Ultimately we now can determine the three-dimensional density distribution with a given Ray-Sum.

However, we can further simplify the equation derived above. By definition, a Fourier transform is:

\[ g(w, \sigma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-2\pi i (wx + \sigma y)} dxdy \]

We might notice then that the equation derived above is:

\[ R(u, \phi) \]
• A Fourier transform of
• With respect to the u variable at $\sqrt{\omega^2 + \sigma^2}$
• With fixed phi at the arctan(sigma,w)

That is:

$$g(w, \sigma) = \hat{R} \left[ \sqrt{w^2 + \sigma^2}, \arctan \left( \frac{\sigma}{w} \right) \right]$$

This will be referred to as the Basic Thermo in the later sections of this paper. Also note the “hat” above the R is a notation describing the fact that R is a Fourier transform, and the subscript of 1 denotes that the Fourier transform is only in the first variable.

**Related Integral Transform**

In order to introduce another similar transform to that of the Fourier Transform, we must first look at the Bessel Function. This is Bessel’s differential equation, which is a second order linear equation with variable coefficients:

$$x^2 y'' + xy' + \left(x^2 - n\right)y = 0$$

The Bessel Function can also be expressed as a series solution. A solution where y equals J sub n of x as shown below:

$$y = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{x}{2} \right)^{n+2k}}{k!(n+k)!}$$
The Bessel Function can satisfy many different equations. The one that we are going to focus on is a formula with integral representation. This formula is integrated from zero to two pi and is a function of order n.

\[ J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{(\sin x + ix \cos s)} ds \]

Now that we have defined the Bessel Function we can take a look at a similar transform that is used in tomography. The Hankel Transform occurs quite naturally in two-dimensional Fourier analysis problems. The basic notation of the Hankel Transform is:

\[ F_n(t) = \int_0^\infty f(x) J_n(tx) x dx \]

This transform is also very similar to the Laplace Transform. The main difference is in the colonels. The Hankel Transform uses the Bessel Function, as the Laplace is exponential. Like the Fourier Transform, the Hankel Transform has an inverse. The inverse and the basic notation look very similar. The two parts are just inverted.

\[ f(x) = \int_0^\infty F_n(t) J_n(tx) dt \]

**Part V: Fourier Series & Polar Form**

In part three of this paper, the Fourier Slice Theorem was derived. The Theorem was expressed using an equation in rectangular coordinates. We will now show how the polar version of the Fourier Slice Theorem can be expressed using the Hankel transform and Fourier Series. To begin with, however we need to introduce the Fourier Series.
A Fourier Series allows any function to be expressed as an infinite sum of complex exponentials (essentially trig functions) multiplied by Fourier Coefficients. The Fourier Coefficients are obtained by taking integrals of the original function.

To start with, we find the Fourier Coefficients for the Fourier Series for the Fourier Transform in the $u$ variable of the ray-sums.

$$\hat{R}_1(t,\phi) = \sum_{n=-\infty}^{\infty} c_n(t)e^{in\phi}$$

where

$$\hat{c}_n = \frac{1}{2\pi} \int_0^{2\pi} \hat{R}_1(t,\phi)e^{-in\phi} d\phi$$

We then take a version of the Fourier Slice Theorem transformed into polar coordinates:

$$f(r,\theta) = \int_0^{2\pi} \int_0^\infty \hat{R}_1(t,B+\theta)e^{2\pi irt \cos B} dtdB$$

We then substitute the Fourier series version of the Fourier transform of the ray-sums for the Fourier transform of the ray-sums.

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty \hat{c}_n(t)d_n(rt)dt$$

$$d_n(rt) = \int_0^{2\pi} e^{int+2\pi rts} ds$$

Using the definition of the Bessel function (found earlier in the paper), we are able to simplify the equation. We are then able to get an equation for the Fourier coefficients of $f$ in polar coordinates:

$$2\pi \int_0^\infty \hat{c}_n(t)J_n(2\pi rt)dt$$
We now have a way of determining $f$ as a Fourier series, with the Fourier coefficients given as $i^n$ times the $n^{th}$ Hankel transform of $\hat{c}_n$. 
Bibliography


Solomon, Frederick. *Three-Dimensional Image Reconstruction*.

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