1. Vanishing of line bundle cohomology

1.1. Notation. Let $G$ be a simple simply connected algebraic group scheme over a field $k$ of characteristic $p$. Let $T$ be a maximal split torus and $\Phi$ be the root system associated to $(G, T)$. The positive (resp. negative) roots are $\Phi^+$ (resp. $\Phi^-$), and $\Delta$ is the set of simple roots. Let $B$ be a Borel subgroup containing $T$ corresponding to the negative roots and $U$ be the unipotent radical of $B$. For $J \subseteq \Delta$, let $P_J \supset B$ denote the corresponding standard parabolic subgroup and $U_J \subset P_J$ denote its unipotent radical. Let $u_J = \text{Lie}(U_J)$ be the Lie algebra of $U_J$. For a rational $G$-module $M$, $M^*$ denotes the $k$-linear dual of $M$.

Let $E$ be the Euclidean space associated with $\Phi$, and the inner product on $E$ will be denoted by $\langle \ , \ \rangle$. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \ldots, \omega_n$. Let $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$. Let $\Xi$ be the Euclidean space associated with $\Phi$, and the inner product on $\Xi$ will be denoted by $\langle \ , \ \rangle$. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \ldots, \omega_n$. Let $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$. Over characteristic zero, these conditions hold by Grauert-Riemenschneider vanishing. However, in prime characteristic, they are not known in general. In the special case $u_J = u = \text{Lie}(U)$ ($J = \emptyset$), condition (1.2.2) was verified by Andersen and Jantzen $[AJ]$ for certain $\lambda$ if $p$ is larger than the Coxeter number and in general by Kumar, Lauritzen, and Thomsen for good primes.

Christophersen $[C]$ has recently developed a strategy for verifying condition (1.2.2) for a specific weight $\lambda$. In her work, she is interested in the case that $\Phi$ is of type $E_6$.

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but much of the theory works more generally. Let $V \subseteq u$ be a $B$-subrepresentation of $u$, $\lambda \in X(T)$, and $i_0 \in \mathbb{Z}$. Suppose one wants to show that

\[(1.2.3) \quad R^i \text{ind}_B^G S^n(V^*) \otimes \lambda = 0 \text{ for all } i > i_0 \text{ and } n \in \mathbb{Z}.\]

Consider the short exact sequence

\[0 \to V \to u \to W \to 0.\]

In [C, Example 3.15], it is shown that condition (1.2.3) holds if the following condition holds:

\[(1.2.4) \quad \text{for } 0 \leq j \leq \dim W \text{ and all weights } \mu \text{ of } \Lambda^j W^* \otimes \lambda, \text{ we have } m(\mu) \leq j + i_0.\]

Here $m : X(T) \to \mathbb{Z}$ is a certain vanishing function introduced in [C, Definition 3.12] (see also [C, Definition 3.6]).

Note that if condition (1.2.4) fails, this does not necessarily imply that condition (1.2.3) fails.

1.3. Programs. For $\Phi$ of type $E_6$, Christophersen [C] created a computer program to verify condition (1.2.4) which implies condition (1.2.3). Taking $V = u_J$, this could potentially verify condition (1.2.1) or condition (1.2.2) for a specific $\lambda \in X(T)$.

Working from the original Java code in [C], undergraduate students at the University of Wisconsin-Stout have modified the program to potentially verify condition (1.2.4) for root systems of arbitrary type. In addition, the code has been rewritten in the form of a Java applet for convenient access on the internet. The program can be found at http://faculty.uwstout.edu/bendelc/research/student.html.

The bulk of this process has been done by Jason Mankovecky with some initial work by Benjamin Mandler and Hannah Rosenthal.

1.4. Instructions. If you wish to test the program, you may click on the Java Applet link on the web page and proceed as below.

If you wish to use this for serious computation, please download all seven files on the web page: one .html file and six .class files. Place them in the save directory. Then execute the Run.html file. This will open a Java applet.

Once the Java applet is open, one can choose the root system of interest. Then the user needs to input three things: the integer $i_0$ (default is zero), the weight $\lambda$ (default is the zero weight), and the weights of $W^*$. Of key importance here is that the weight $\lambda$ and the weights of $W^*$ must be expressed as positive integral linear combinations of simple roots. Since $W^*$ necessarily has basis consisting of positive roots, this is not a restriction for these weights. However, this does place a restriction on the weights $\lambda$ which can currently be tested. Note that the program involves a potentially large number of operations, particularly as the rank and number of weights increase. So one should be prepared for the program to potentially run for a significant length of time.

Example. Suppose one wants to verify condition (1.2.2) in type $E_7$ with $J = \{\alpha_1, \alpha_3\}$ (using Bourbaki notation) and $\lambda = 2\alpha_4 + \alpha_7$. Then $W = u/u_J$ and $W^*$ is three
dimensional with a root basis corresponding to $\{\alpha_1, \alpha_3, \alpha_1 + \alpha_3\}$. To run the program, one would first choose type $E_7$-Bourbaki. Then one needs to click the add weight button twice to allow for three weights in $W^*$. One can then input the value of $i_0 = 0$ which will be the default. Next, express $\lambda = (0, 0, 0, 2, 0, 1)$ in vector form as a linear combination of simple roots. One simply inputs these numbers in the boxes. Similarly, under the boxes labeled “Weights” one would submit $(1, 0, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0, 0)$, and $(1, 0, 1, 0, 0, 0, 0)$ in any order. Finally, click on calculate. The response will be either “True”, meaning that condition (1.2.4) holds, or “False”, meaning that condition (1.2.4) fails.

REFERENCES


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