Origins and Interpretations of the Concept of Convolutions

Presented By

Craig Rykal
John Rivera
Justin Malaise
Jeff Swanson
Ben Rougier

From

University of Wisconsin STOUT

Under the direction of Dr. Steve Deckelman

Abstract

The convolution of two functions is an important concept in a number of areas of pure and applied mathematics such as Fourier Analysis, Differential Equations, Approximation Theory, and Image Processing. Nevertheless convolutions often seem unintuitive and difficult to grasp for beginners. This project explores the origins of the convolution concept as well as some computer graphic and physical interpretations of convolution which illustrate various ways the technique of smoothing can be used to solve some real world problems.
Convolutions

\[ F(t) = \int_{\mathbb{R}} f(x)k(t - x) \, dx \]

Convolutions can be thought of as a method of averaging unruly functions.

Unruly Functions include:

- Functions with sharp or jagged edges
- Discontinuous functions
Origin of Convolutions

Weighted Averages \[ \sum_{j=1}^{n} \lambda_j y_j \]

Substitute \( y = f(x) \) into the function to receive \( \sum_{j=1}^{n} \lambda_j f(x_j) \)

Now we need \( \sum_{j=1}^{n} \lambda_j f(x_j) \) to return a function

Therefore we create a new function

\[ \sum_{j=1}^{n} \lambda_j f(x - x_j) \]

Which returns a function
One more substitution of $\lambda_j = g_j(x)$

Gives us the function:

$$\sum_{j=1}^{n} g_j(x) f(x - x_j)$$

This function will give us a new discrete function, but we need this function in continuous form.

To accomplish this, change the function to an integration.

This gives us:

$$\int_{R} g(t) f(x - t) \, dt$$

Which is a Convolution!
Examples

Take the function

\[ f(x) = \begin{cases} 
0 & \text{for } x < -h \\
1 & \text{for } -h \leq x \leq h \\
0 & \text{for } h > x 
\end{cases} \]

Convolve the function with itself to receive the new function

\[ F(t) = \begin{cases} 
0 & \text{for } t < -2h \\
t + 2h & \text{for } -2h \leq t < 0 \\
2h - t & \text{for } 0 \leq t \leq 2h \\
0 & \text{for } 2h < t 
\end{cases} \]
One more convolution gives us the continuous function

\[ G(t) = \begin{cases} 
0 & t < -3h \\
(t + 3h)^2 / 2 & -3h \leq t < -h \\
-t^2 + 3h^2 & -h \leq t < h \\
(t - 3h)^2 / 2 & h \leq t \leq 3h \\
0 & 3h < t
\end{cases} \]

Convolutions are also useful in smoothing functions of more than one dimension

\[ F(s, t) = \iint_{R \times R} f(x, y)k(s - x, t - y) \, dx \, dy \]
Antialiasing

Aliasing
Eratosthenes

230 B.C
Alexandria, Egypt

\[ \alpha = 7.2^\circ \]
Length(\(A\)) = 787 KM

\[ r = \frac{c}{2\pi} \]

\[ c = \frac{767 \times 360^\circ}{7.2^\circ} \]

\[ r = 6103.59 \]
Parallax - any alteration in the relative apparent positions of objects produced by a shift in the position of the observer.

$R =$ radius of Earth.

$\theta =$ angle between observation points with relation to the center of the Earth.

P & Q are observation points on the Earth.

$X =$ distance between the center of the sun and the center of the Earth.

**How To find the Distance:**

If we know $|PQ| \Rightarrow |PQ| = R \theta \Rightarrow \theta = |PQ| / R$

$\pi/2 - \theta =$ angle of center of sun to P & Q

$R/X = \cos (\theta) \Rightarrow X = R/\cos(\theta) = R(\sec(\theta))$

The only problem is that there must be a very accurate measurement of $\theta$ in order to get an accurate distance. Hipparchus (130 BC) and Ptolemy (150 AD) used the value of the diameter of the earth given by Eratosthenes (195 BC) and estimated the distance to be 10 million miles. We know that the distance is 93 million miles away. Thus, even close but not precise values give huge errors.
The Diameter of a Star

WE SEE:

TRUE IMAGE(shape):

$$f(x) = \begin{cases} \lambda & (x \text{ inside disc}) \\ 0 & \text{otherwise} \end{cases}$$
\[ f(x) = \lambda \ D\left( \frac{x - x_0}{\varepsilon} \right) \]

Note: \( \lambda, \ x_0, \ \varepsilon \) are all unknown quantities.

Where \( D(x) \) is the characteristic function of the unit circle.

\[
D(x) = 1 \quad \text{for} \ |x| \leq 1, \]

\[
D(x) = 0 \quad \text{for} \ |x| > 1
\]

**Fundamental Problem:**
Calculate \( \varepsilon \) from \( f \)
Let

\( \lambda \): “Brightness” at point source O
\( \lambda' \): “Brightness” at point source Y
\( \lambda K_t(X) \): “Brightness” at X
\( \lambda' K_t(X - Y) \): “Brightness” at X arising from point source Y.
\( K_t(X) \): Apparent Brightness
True Brightness (At time t)
We Get:

\[ A(x) = \text{"Brightness at } X\" = \sum_{j=1}^{n} \lambda_j K_t(X - Y_j) \]

\( Y_1, Y_2, \ldots, Y_n \rightarrow \) “Brightness” of \( Y_j \) given point sources

\( ||Y||_1, ||Y||_2, \ldots, ||Y||_n \rightarrow \) Point sources of light

Given are very small

Let’s pass from discrete to the continuous model:

\[ \sum_{j=1}^{n} \lambda_j K_t(X - Y_j) \rightarrow \iiint f(Y) K_t(X - Y) \, dA(Y) \]

\( f(Y) \): Brightness at point \( Y \). \( K_t(X - Y) \): Blurring effect(kernel) of the atmosphere at time \( t \).

\( dA(Y) \): Integrate with respect to \( Y \).

TRUE IMAGE(brightness) = \( f(x) \)

ACTUAL IMAGE = \( f * K_t \)
1 a.u. = distance from the sun to the earth

$\theta$ is found experimentally

$d = \sec(\theta)$
Knowing $d$ and $\mathcal{E}$ will allow us to find the diameter of the star.

The Fundamental Problem:

Extract $f$ from $f^*K_t$ where $K_t$ is an unknown random function.

Labeyrie’s Idea:

Use Averaging and Fourier Transforms
1. Averaging $K_t$:

To get a fixed $K_t$ or the “Average Blur”:

Average the image received at various times, $t(1), t(2), ..., t(n)$

$$\tilde{K} = \sum_{j=1}^{n} \frac{1}{n} K_{t(j)} : \text{AVERAGE BLUR}$$

Average the Convolutions, $$\frac{1}{n} \sum_{j=1}^{n} f * K_{t(j)} = f * \sum_{j=1}^{n} \frac{1}{n} K_{t(j)} = f * \tilde{K}$$

2. Using the Convolution Theorem for Fourier Transforms:

$$\phi_t = f * K_t : \text{FOURIER TRANSFORM of our image(convolution)}$$

$$\hat{\phi}_t = \hat{f} * \hat{K}_t : \text{Under the Convolution Theorem for Fourier Transforms}$$
Take the sequence: \( \hat{\phi}_{t(1)}', \hat{\phi}_{t(2)}', \hat{\phi}_{t(3)}', \ldots', \hat{\phi}_{t(n)} \)

\( \hat{\phi}_{t(j)} \) is RANDOM, due to \( K_t \) being RANDOM.

The only zeros (roots) the \( \hat{\phi}_{t(j)} \) 's will have in common with all other \( \hat{\phi}_{t(j)} \) 's

Are the zeros of \( \hat{f} \) !

Superimposing \( \left| \hat{\phi}_{t(j)} \right| \)

forms \( \Omega(\zeta) = \sum_{j=1}^{n} \left| \hat{\phi}_{t(j)}(\zeta) \right| = \sum_{j=1}^{n} \left| \hat{f}(\zeta) \right| \left| K_{t(j)} \right| \)

Clearly the zeros of \( \hat{f}(\zeta) \)

Will stand out as the zeros of \( \Omega(\zeta) \)
\[ \hat{f}(\zeta) = \int_\mathbb{R}^2 \int_\mathbb{R}^2 \lambda \ D \left( \frac{x - x_0}{\varepsilon} \right) \ e^{-2\pi i \zeta x} \ dA(x) \]

Substitute \( y = x - x_0 \)

\[ \hat{f}(\zeta) = \int_\mathbb{R}^2 \int_\mathbb{R}^2 \lambda \ D \left( \frac{y}{\varepsilon} \right) \ e^{-2\pi i (y + x_0)} \ dA(y) \]

Move out the constants

\[ \hat{f}(\zeta) = \lambda \ e^{-2\pi i \zeta x_0} \int_\mathbb{R}^2 \int_\mathbb{R}^2 D \left( \frac{y}{\varepsilon} \right) e^{-2\pi i \zeta y} \ dA(y) \]

Substitute \( w = y/\varepsilon, \ y = \varepsilon w \)

\[ \hat{f}(\zeta) = \lambda \ \varepsilon^2 \ e^{-2\pi i \zeta x_0} \int_\mathbb{R}^2 \int_\mathbb{R}^2 D(w) e^{-2\pi i \zeta \varepsilon w} \ dA(w) \]

By the definition of Fourier Transforms

\[ \int_\mathbb{R}^2 \int_\mathbb{R}^2 D(w) e^{-2\pi i \zeta \varepsilon w} \ dA(w) = \hat{D}(\varepsilon \zeta) \]
\[ \hat{f}(\zeta) = \lambda \, \epsilon^{2} \, e^{-2\pi i \zeta x_0} \hat{D}(\epsilon \zeta) \]

We see that the zeros of \( \hat{f}(\zeta) \) are the same as the zeros of \( \hat{D}(\epsilon \zeta) \).

To compute \( \epsilon \), we need to know the location of the rings of zeros of \( \hat{D}(\epsilon \zeta) \).

The zeros occur on the circle of radius \( \zeta = r_0 \), i.e.
\[ \hat{D}(\epsilon \zeta) = 0 \quad \text{When} \quad \zeta = r_0 \]

Which should be visually evident from our \( \Omega \) picture.

\[ \hat{D}(\zeta) = \frac{2\pi J_1(\|\zeta\|)}{\|\zeta\|} \]

Where \( J_1 \) is the first order Bessel function

\[ J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \frac{x^9}{2^9 4!5!} - \cdots \]

The zeros occur on the circle of radius \( \zeta = r_1 \), i.e.
\[ \hat{D}(\zeta) = 0 \quad \text{When} \quad \zeta = r_1 \]

\textbf{NOTE:} \( r_1 \) can be found analytically.
So if
\[ \hat{D}(e_{r_0}) = 0 \quad \text{and} \quad \hat{D}(r_1) = 0 \]
then
\[ e_{r_0} = r_1 \]
And thus
\[ e = \frac{r_1}{r_0} \]

Accuracy:

- Labeyrie’s method gives results consistent to Michelson’s interferometer results.
- The method has been applied to over 30 stars already.
References

• “Fourier Analysis” by T.W. Korner, Cambridge University Press, 1988
• “Convolutions and Computer Graphics” by Anne M. Burns, College Mathematics Journal, 1992
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