Catcher of the Fly

By:

Janel Krenz, Lori Naiberg, Beh Rahn, Ivan Lau, and Chad Seichter

Models I
Introduction
Our group has extensively studied the paper “Mathematician catches a baseball” by Edward Aboufadel. In this paper, he suggested that the Strong LOT model, which is based on the LOT (Linear Optical Trajectory) model by Michael McBeath, Dennis Shaffer and Mary Kaiser, provides a mathematical foundation on which the LOT model can rest. He also developed equations relating the motion of a fly ball to the motion of an outfielder utilizing the LOT strategy.

Background
There have been a lot of studies done to try and figure out how an outfielder figures out where to run to catch a fly ball. Michael McBeath, Dennis Shaffer, and Mary Kaiser came out with a model in 1995 called the Linear Optical Trajectory Model more commonly called the LOT model. Before the LOT model, researchers believed that baseball players made complex calculations about the arc, acceleration, deceleration, and distance they were away from the ball. They then solved the problem in three-dimensions. That theory also predicted that the fielder would run a straight path and do this at a constant speed. The new LOT model predicts that the fielder will run a curved path which will result in a linear optical ball trajectory. Fielders using the LOT model are able to break down the three-dimensional problem into a much simpler two-dimensional problem. The ball will constantly be curving, first up and then down. Using the basic LOT model, the outfielder keeps the ball on a straight line in the visual field while the ball is curving to determine where to go.

Aboufadel’s Mathematical Model
The diagram shown below will help as the paper goes on to describe the mathematical equations behind Aboufadel’s Mathematical Model. The diagram is a picture of the points and lines needed for the following proofs. The green (or light gray if you have a black and white copy) shaded area represents the ground and the blue (or dark gray if you have a black and white copy) shaded area represents the planes that are perpendicular to the ground.
The points all have meaning and components, which will give a better understanding of the diagram. The coordinates of the F, B, and I points are important in the proofs.

“H” represents home plate and is where the ball is hit from. “H” has coordinates (xh, yh, zh). “H” is also the point chosen as the origin and will always be on the ground, therefore the coordinates are (0, 0, 0).

“F” is the position of the fielder and has coordinates (xf, yf, zf). The fielder will also always be on the ground so its z component will be zero giving “F” coordinates of (xf, yf, 0).

“B” is the position of the ball and has (xb, yb, zb) as its coordinates.

“B*” is the projection of the ball onto the field. It’s coordinates are not used in the calculations.
“I” has coordinates \((x_i, y_i, z_i)\) and is the fielder’s image of the ball.

“I*” is the unique perpendicular and it’s coordinates are also not used.

There are two other functions that are defined to be used in the calculations: \(p = y_i/x_i\) and \(q = z_i/x_i\)

All of the coordinates are functions of time \(t\) with \(t = 0\) representing the time when the batter hits the ball.

Aboufadel derived the Strong LOT Model, which is a special case of the LOT model that was discussed previously. The Strong LOT Model is different in that is has the hypothesis: The strategy that the fielder uses to catch a fly ball is to follow a path that keeps both \(p\) and \(q\) constant. To remind you, \(p = y_i/x_i\) and \(q = z_i/x_i\).

It is also important to remember through the rest of the paper that the fielder \(F = (x_f, y_f, z_f)\), the ball \(B = (x_b, y_b, z_b)\), and the fielder’s image of the ball \(I = (x_i, y_i, z_i)\).

The Strong LOT model hypothesis states: The strategy a fielder uses to catch a fly ball is to follow a path that keeps both \(p\) and \(q\) constant.

For this hypothesis, the line \(HI^*\) has a slope \(p\), \(B^*F\) has a slope of \(-1/p\) because \(B^*F\) is the perpendicular to \(HI^*\). Using the definition of slope, we find

\[
-\frac{1}{p} = \frac{y_f - y_b}{x_f - x_b} \implies p = \frac{x_b - x_f}{y_f - y_b}
\]

Equation (3)

and (3) is true at every point in time.
The equation of $HI^*$ is $y = px$ and the equation for $B^*F$ is $y = y_b - (x - x_b)/p$ and the point $I^*$ determined by the intersection of these two points. We set them equal to each other and solve.

\[
px = y_b - \frac{(x - x_b)}{p} \Rightarrow
\]

Set them equal, subtract $y_b$ from both sides and multiply both sides by $p$:

\[
p^2x - y_b p = -x + x_b \Rightarrow
\]

Add $x$ and $y_b p$ to both sides:

\[
p^2x + x = x_b + y_b p \Rightarrow
\]

Factor out an $x$ from the left side:

\[
x(p^2 + 1) = x_b + y_b p
\]

Divide both sides by $(p^2 + 1)$ and we get:

\[
x_i = \frac{x_b + y_b p}{p^2 + 1}
\]
Therefore we conclude $x_i$, the x-coordinate of the point $I^*$, is the intersection of these the two lines $HI^*$ and $B^*F$.

Now, since we know that F, B, and I are collinear, we have:

$$\frac{z_i - z_f}{x_i - x_f} = \frac{z_b - z_f}{x_b - x_f}$$

Since $z_f = 0$, because the fielder is on the ground at all times, and also $z_i = qx_i$, plug these values into the equation above and solve for $x_f$.

$$\frac{qx_i}{(x_i - x_f)} = \frac{z_b}{(x_b - x_f)} \Rightarrow$$

Cross Multiply:

$$qx_i x_b - qx_i x_f = z_b x_i - z_b x_f \Rightarrow$$

Subtract $qx_i x_b$ from both sides and add $z_b x_f$ to both sides:

$$z_b x_f - qx_i x_f = z_b x_i - qx_i x_b \Rightarrow$$

Factor out the $x_f$:

$$x_f (z_b + qx_i) = z_b x_i - qx_i x_b \Rightarrow$$

Divide by $z_b + qx_i$ and we get:

$$x_f = x_i \left( \frac{z_b - qx_b}{z_b - qx_i} \right)$$
Next, we combine equation 4 and 5 to get the mathematical expression for the x-coordinate of the fielder.

\[ x_f = \frac{x_b + p y_b}{p^2 + 1} \left( \frac{z_b - qx_b}{z_b - qx_i} \right) \]

Substitute in \( x_i \) from equation (4) into equation (5):

\[ \frac{(z_b - qx_b)(x_b + p y_b)}{(p^2 + 1)\left(z_b - q\left(\frac{x_b + p y_b}{p^2 + 1}\right)\right)} \]

Multiply through and solve for \( x_i \):

\[ x_f = \frac{(z_b - qx_b)(x_b + p y_b)}{z_b(p^2 + 1) - q(x_b + p y_b)} \]

This gives the expression for the x-coordinate of the fielder.

Now we will find the y-coordinate of the fielder. First we solve the equation (3) for \( y_f \).

\[ p = \frac{x_b - x_f}{y_f - y_b} \]

Multiply both sides by \( y_f - y_b \):
Add $py_b$ to both sides and divide both sides by $p$:

$$y_f = \frac{x_b - x_f + p y_b}{p}$$

Combining $y_f$ and equation (6) and solving we get:

$$y_f = \frac{(pz_b - qy_b)(x_b + p y_b)}{z_b (p^2 + 1) - q(x_b + p y_b)}$$

Equation (7)

This gives us the mathematical expression for the y-coordinate of the fielder.

We now solve equation (6) for $q$ and we get:

$$q = \left( \frac{z_b}{x_b + p y_b} \right) \left( \frac{x_f (p^2 +1) - (x_b + p y_b)}{x_f - x_b} \right)$$

Equation (8)

We now have, for every $t > 0$ and for every trajectory $B$, a relationship between $(x_f, y_f)$ and $(p, q)$. If we know what $p$ and $q$ are, then equations (6) and (7) give us the $x$ and $y$ positions of the fielder and if we know the $x$ and $y$ positions of the fielder, then we can derive $p$ and $q$ by equations (3) and (8).

The $x$ and $y$ positions of the fielder were proved in the previous proofs. It is now important to prove that the fielder will intersect with the ball. Time $t=T$ is defined to be
when the ball hits the ground. Equation 6 (the x coordinate of the fielder) is used, which was proved earlier. When \( t=T \), the ball hits the ground so the z coordinate of the ball will be 0. Substitute 0 in for z and the second equation follows. The quantity in the parentheses can then be cancelled and \( x_b \) remains. This proves that at \( t=T \), when the ball hits the ground, \( x_f = x_b \) so the x position of the fielder intersects with the x position of the ball. It is also proved in the same manner that \( y_f = y_b \).

\[
\left. x_f \right|_{t=T} = \left. \frac{(z_b - qx_b)(x_b + py_b)}{z_b(p^2 + 1) - q(x_b + py_b)} \right|_{t=T}
\]

\[
= \left. \frac{-qx_b(x_b + py_b)}{-q(x_b + py_b)} \right|_{t=T}
\]

\[
= x_b \left|_{t=T} \right.
\]

As a consequence of the Strong LOT model, since \( p \) and \( q \) are constant, you can calculate them. We will use previous equations and show the calculations necessary to come up with the equations for \( p \) and \( q \).

Since equation 3 (which determines the slope of the line \( HI^* \)) is true for all \( t \), it is also true for when the batter hits the baseball (\( t=0 \)), therefore the following holds:

\[
p \left|_{t=0} = \frac{x_b - x_f}{y_f - y_b} \right|_{t=0} = \left. -\frac{x_f}{y_f} \right|_{t=0}
\]

Explanation: When \( t=0 \) the x and y position of the ball are zero, therefore through cancellation of the \( x_b \) and \( y_b \) terms you get the above equation.
To determine q, we use equation 8 from previous calculations and L’Hopital’s Rule.

We will take the limit of equation 8 as t tends to zero and get the following:

\[
q \bigg|_{t=0} = \lim_{t \to 0} \frac{z_b}{x_b + py_b} \left( x_f \left( p^2 + 1 \right) - (x_b + py_b) \right) \frac{x_f - x_b}{x_f - x_b}
\]

When t=0 the x and y position of the ball are zero, therefore we will let xb and yb equal zero in the second part of the above equation.

The xf from the numerator and denominator will cancel. Therefore you end up with the following:

\[
= \lim_{t \to 0} \frac{z_b}{x_b + py_b} \left( p^2 + 1 \right)
\]

We will finally apply L’Hopital’s Rule to the above equation. Taking the derivative of the numerator and denominator, we get the following:

\[
= \frac{z'_b \left( p^2 + 1 \right)}{x'_b + py'_b} \bigg|_{t=0}
\]

Unanswered Questions

In the Strong LOT model, we assumed that p and q were constants and we showed that the ball would intersect with the field. The Question is what if p and q are not constants? In his paper, he did not explain why we had to assume p and q were constants in order to get a unique path. If p and q were constants, could we find more than 1 path for the fielder to catch the ball? If there were more than one path, could we find a shorter path than the Strong LOT model provided?

If the fielder establishes a strong LOT model, can he/she ran straight to the destination point and wait there rather than chasing after the ball? The Strong Lot model did not say
how fast the fielder had to run relative to the speed and position of the ball. What is the “ideal speed” for the fielder to run on the path from strong LOT model?

**Competing Models**

One of the competing models to the Strong LOT Model is the OAC or optical acceleration cancellation model. The OAC model predicts that fielders select a straight running path and keeps a constant speed while retrieving the ball. One of the main arguments about this model is that it requires accurate calculation of the ball’s acceleration which people normally have trouble doing. Another problem with this model is that since it maintains lateral alignment, the balls location will be a planar optical projection even though in real life, the projection is spherical.

Another competing model is Adair’s Model. Adair’s Model focuses on the path of the ball, such as the curvature. He suggested that a fielder runs laterally so that the ball goes straight up and down from his or her view. However, people argued that both maintenance of lateral alignment and monitoring of up and down ball motion require information that is not perceptually available from the fielder’s vantage.

**Conclusion**

As you can see, there are many different models one can follow when trying to catch a fly ball. Our group extensively studied the Strong LOT model. This is a special case of the original LOT model. In the Strong LOT model, p and q must be constant. If you know p and q, you can calculate the x and y coordinates of the fielder.

In our studies, we took a look at other competing models such as the Optical Acceleration Cancellation (OAC) Model and Robert Adair’s Model. These models take a very different look at catching a fly ball than the Strong LOT model. It seems all of the models have some unanswered questions.

Our group has learned a lot about the mathematics behind catching a fly ball. It may be hard to make these complex calculations when on the field, but if you could, you may never miss a fly ball again.
Bibliography


